SOLVABILITY AND UNIQUENESS OF SOLUTIONS OF TWO DIMENSIONAL OPTIMIZATION PROBLEMS

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Abstract:

This paper examines the 2 – dimensional optimization theory, its problems and solutions. Lots of work exist in literature studying the one - dimensional optimization theory and its problems / solutions. Hence, our attention is squarely on 2-dimensional optimization theα^0ory and its problems but we are often challenged by these questions - are they solvable and if the response is in the affirmative, then are the solutions produced unique? In the course of our study, we established that the 2-dimensional optimization problem is solvable, that is a 2 – dimensional optimal control problem is solvable, by providing proofs to some logical assertions. Furthermore, we also showed that the solutions generated from these problems - the optimal control problems are unique and prove the optimality of the solution obtained by checking $J(\alpha) \geq J(\alpha^0)$ for any$\alpha \in G_T(V)$. These findings provide and encourage avenues for the application of 2 – dimensional optimization models in the industrial sector.

Keywords: 2 – dimensional problems, solvable, unique solution, linear processes, optimality, industrial sector

Introduction:

In literature, there exists several studies of the 1-dimensional optimization theory and problems in both discrete and continuous cases. Aderibigbe (1993) constructed an operator that enabled the implementation of the algorithm for the extended conjugate gradient method (ECGM) for control systems with delay – I. In this paper, the author used the 1- dimensional optimization problem – continuous case to execute his work. Otunta (1991) in studying optimization techniques approached the construction of an operator to enable the implementation of the ECGM from a different point of view. Otunta (2003) developed a gradient method for Discrete Optimal Control Problems using a 1 – dimensional optimization Problem. Apanapudor, *et al* (2020) developed an expression that will enhance the computation of an optimal penalty constant and thus avoid indiscriminant selection of penalty constant in solving a constrained optimization problem using the ECGM. In developing this expression, the author, employed the 1– dimensional optimal control problem – the discrete case. Similarly, Aderibigbe and Apanapudor (2014) worked on the ECGM algorithm for Discrete Optimal Control Problems and some of its features. Adebayo and Aderibigbe (2014) combined the Bolza form and Meyer form of the optimal control problem and constructed an operator that enabled them solve one dimensional optimal control problem with delay using the Extended Conjugate Gradient Method (ECGM). In discussing graph theory, Papernov (1976) talked about the feasibility of multi-commodity flows, in Discrete Optimization and Apanapudor, et al (2023) discussed the generalized minimum flow problem with the intent of application in natural gas distribution networks; Mrad and Haouari (2008) investigated the multi-commodity network design, in which a discrete set of skills with mesh-increasing cost and functions were installed on each edge. The design has lots of applications in contemporary telecommunication networks. West (1996), also enumerated several definitions of concepts that are both useful in graph theory and optimization. Izevbizua and Apanapudor (2019, 2020) studied fixed lifetime Inventory Models to changes in costs in diverse means. Earlier in (2018) worked on fixed lifetime Inventory System in two orders and stochastic demand. Apanapudor and Tsetimi (2020) examined the Stochastic Bellman using the conditional certainty property expectations while considering systems of different dimensions. Omokoh, et al (2023) considered some forms of applicatios by using R-programming to stimulate some computational results Furthermore, Bose (2017), in discussing possible needs for application of n – dimensional models in systems theory, opined that there are often bottlenecks in migrating from 1 – dimensional to 2– dimensional as well as to n – dimensional systems. The author, remarked that these challenges and the promised rewards, have shut up this area of study to a state of maturity that guarantees a continuing proliferation of increasingly complex and diversified nature of activities in the area. Gerhard (2002) considered 2-dim systems of Fornasini-Marchesini type that are continuous time cases with varying coefficients. In an attempt to represent the solutions of these systems explicitly, using Riemann Kernel of the equation under consideration, the author obtained some useful criteria in the case of inhomogeneous equation. Nam and Rougerie (2020) studied the ground – state energy of N attractive bosons in plane. The study dwelt on gas to be diluted, so that the corresponding mean-field problem is a local non-Schrodinger (NLS) equation. Franca, et al (2017) using the knowledge based on the two body S-matrix, studied 2-dimensional Bose and Fermi gases with both positive and negative interactions. Their study led to the successful recovering of the correct functional form of the critical chemical potential and density for the Bose gas.

Materials and Methods

We present in this section some useful fundamental concepts, definitions and notations that will enhance our understanding of this paper. Let us begin with the concept of inner product, which is an extensive study of the notion of dot product (vector multiplication). Recall that when we multiply vectors together, the result is a scalar.

Let's assume *s*, *w* and *v* be vectors in a real vector space V and λ be a scalar from the field of R (real numbers). Let's also borrow the notion of Cartesian sets. The set $A \times B$, is called the Cartesian product of sets A and B. Be reminded too, of the notion of mapping. Formally, we may define our concept of inner product over vector spaces as

Definition 2.1. Inner Product

This is a function that associates a real number $\langle s, w\rangle$ with each vector *s* and *w* in a vector space V. Alternatively,

The mapping $\langle s, w\rangle : V \times V \to R$ is called an inner product over *V* if the following axioms are satisfied for $s, w, v \in V$ and all scalars λ,

i) $\langle s, w\rangle = \langle w, s\rangle$,

ii) $\langle s + w, v\rangle = \langle s, v\rangle + \langle w, v\rangle$,

iii) $\langle \lambda s, w\rangle = \lambda\langle s, w\rangle$

iv) $\langle s, s\rangle \geq 0$, for any $s \in V$ and $\langle v, v\rangle = 0$, if and only if $s = 0$.

Hence *V*, the vector space, together with the inner product$\langle s, w\rangle$, is called an inner product space. That is the pair $(V, \langle .,.\rangle)$, is called an inner product space.

Examples are:

(1) R^n, a vector space with the map $\langle s, w\rangle$ defined for arbitrary vectors, by

$$s = (s_1, s_2, \ldots, s_n), w = (w_1, w_2, \ldots, w_n)$$

$$\langle s, w\rangle = \sum_{i=1}^{n} s_i w_i \qquad (2.1)$$

(2) $\mathbb{C}^n$, a complex vector space with the function $\langle s, w\rangle$, defined for arbitrary vectors in $\mathbb{C}^n$ by

$$\langle s, w\rangle = \sum_{i=1}^{n} s_i \varpi_i \qquad (2.2)$$

(3) The vector $V[0,1]$ with the map $\langle s, w\rangle$ defined for arbitrary vectors $f, g \in V[0,1]$ by

$$\langle f, g\rangle = \int_0^1 f(w) g(w) dw$$

(where the "bar" indicates complex conjugation).

Some fundamental properties of inner product space

i) For arbitrary s, w, $v \in V$ and $\lambda, \in F$, then $\langle s, \lambda w + \mu v\rangle = \lambda\langle s, w\rangle + \mu\langle s, v\rangle$, complex conjugation of λ, μ

ii) For arbitrary s, w, $v \in V$ is an inner product space, then

$$|\langle s, w\rangle|^2 \leq \langle s, s\rangle\langle w, w\rangle$$

and equality holds if and only if *s* and *w* are linearly independent.

iii) The function $\|.\| : V \to R$ defined by $\|s\| = \sqrt{\langle s, s\rangle}$, is a norm on *V*.

iv) $\langle .,.\rangle$, is a continuous function on $V \times V$.

Definition 2.2. Cauchy Sequence

A sequence $\{S_n\}_{n=1}^{\infty}$ in V is called Cauchy if and only if

$$\langle x_n - x_m, x_n - x_m\rangle^{1/2} = \|x_n - x_m\| \to 0, \text{ as } n,m \to \infty \qquad (2.3)$$

As a result, an inner product space *V* is called complete if every Cauchy sequence in *V*, converges to a point of *V*. A complete inner product space is called a Hilbert Space and be called a real or complex Hilbert space based on the underlying vector space.

Definition 2.3.

A Hilbert Space is a vector space *V* with an inner product $\langle s, w\rangle$such that the norm defined by $\|s\| = \sqrt{\langle s, s\rangle}$ turns *V* into a complete metric space.

Definition 2.4. *N*-dimensional

A space *X* is said to finite dimensional (*n* – dimensional) if in *X*, there exist a finite basis (basis from *n* – elements).

Definition 2.5. Adjoint Operator

Assume V_1 and V_2 are Hilbert spaces and $\rho : V_1 \to V_2$ be a vector operator (linear operator). We shall call operator $\rho^* : V_2 \to V_1$ adjoint to ρ iff,

$$\langle \rho s, w\rangle_{V_2} = \langle s, \rho^* w\rangle_{V_1} \qquad (2.4)$$

Holds for any $w \in V_2, s \in V_1$ Or

Given a Hilbert space *V* and a map $\rho : D(\rho) \subset V \to V$ that is bounded. Then a map defined as

$$\rho^* : V \to V$$

By the relation

$$\langle \rho s, w\rangle_{V_2} = \langle s, \rho^* w\rangle_{v_2} \qquad (2.5)$$

for all $s \in D(\rho), s \in D(\rho^*)$, is called the adjoint of ρ

We state without proof some fundamentals for adjoint operators on Hilbert spaces in the following theorem.

Theorem 2.1.

Assume ρ, ρ_1, $\rho_2 : V \to V$ be bounded linear operators with adjoints $\rho^*, \rho_1^*, \rho_2^*$ respectively. Then

(i) $(\rho_1 + \rho_2)^* = \rho_1^* + \rho_2^*$

(ii) $(\lambda\rho)^* = \bar{\lambda}\rho^*$, λ is a complex scalar and its conjugate $\bar{\lambda}$

(iii) $(\rho_1\rho_2)^* = \rho_2^*\rho_1^*$

(iv) $(\rho^*)^* = \rho$

(v) $\|\rho\| = \|\rho^*\|$

(vi) $\|\rho\|^2 = \|\rho^*\rho\|$

Definition 2.6.

Suppose $\|\bullet\|_1$ and $\|\bullet\|_2$ be norms in a vector space V. We say that $\|\bullet\|_1$ is equivalent to $\|\bullet\|_2$ if there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\lambda_1 \|w\| \le \|w\| \le \lambda_2 \|w\| \quad (2.6)$$

for all $w \in$ V.

In an n – dimensional Hilbert space, where U, V are spaces, with the inner product defined by $\langle .,. \rangle$. Let $\beta_i: U \to U, i = -k, k+1, \dots, 0, 1, \dots, k; k \in N$ and a linear operator $G: V \to U$. A two dimensional (2 – D) discrete system can be defined and described by

$$w(\gamma+1, y) = \sum_{i=-k}^{k} \beta_i(\gamma, y+1) + G\mu(\gamma, y), \gamma = 0,1,2,\dots, y \in Z. \quad (2.7)$$

Where $\mu(\gamma, y)$ is the V-valued function.

For any function $\rho : Z \to u$, assume a unique solution $w(\gamma, y$which satisfies the initial condition

$$w(0, y) = \rho(y), y \in Z.$$

We shall call $\mu(\gamma, y)$ the controlled factors; for instance kerosene pressure and kerosene flow at particular points in a given kitchen pipe needed to maintain the desired regime. For a linear map $\beta_i: U \to U, i = -k, k+1, \dots, 0, 1, \dots, k; k \in N$, define the operator $\beta : l^2(u) \to l^2(u)$ as

$$(\beta\vartheta)(y) = \sum_{i=-k}^{k} \beta_i \vartheta(y+i), y \in Z \quad (2.8)$$

Definition 2.7.

A conjugate operator for the operator β is the map β^* $\beta^* : l^2(u) \to l^2(u)$ Defined by

$$(\beta^*\psi)(y) = \sum_{i=-N}^{N} \beta_i^* \psi(y-i), y \in Z \quad (2.9)$$

Where β^* is the conjugate operator β_i.

Using equation (2.9) we, define the conjugate equation of (2.10) as

$$z(\gamma, y) = \sum_{i=-k}^{k} \beta_i z(\gamma+1, y-i) + g(\gamma, y), \quad (2.10)$$

Where $z(\gamma, y)$ is an unknown function.

It is necessary we present equation (2.7) and (2.9) in an operator form. To achieve this, let

$$y \to u(\gamma, y) \to u(\gamma, y); y \to g(\gamma, y)$$

Be squared for each fixed $y, y \ge 0$. Also let be in $K_\gamma, \vartheta_\gamma, \psi_\gamma$ $l^2(u)$ space, $\mu_\gamma$be in $l^2(\alpha)$ space defined by

$$(K_\gamma)(y) = w(\gamma, y), (\vartheta_\gamma)(y) = z(\gamma, y), (\psi_\gamma)(y) = g(\gamma, y), (\alpha_\gamma)(y) = u(\gamma, y), \gamma \ge 0, y \in Z \quad (2.11)$$

Hence we write equations (2.7) and (2.9) as follows

$$\kappa_{\gamma+1} = \beta\kappa_\gamma + G\alpha_\gamma$$
$$\vartheta_\gamma = \beta^*\vartheta_{\gamma+1} + \psi_\gamma \quad (2.12)$$

Where G is the operator from $l^2(V)$ and $l^2(U)$ defined by $(\beta\kappa)(y) = \beta\theta(y), y \in Z$. We shall employ this operator form in proving the solvability of optimization problems in our subsequent section. The γ following are the main theorems we shall address and prove this paper.

Main Theorem:

A: An optimization problem as presented in equations (2.7) and (3.1) is solvable.

B: The optimal control problem in (2.7) and (3.1) possesses a unique solution

$$\mu^0(\gamma, y) = -R^{-1}G^* z(\gamma, y), \gamma \epsilon [0,1,\dots,T], y \epsilon Z \quad (2.13)$$

where $z(\gamma, y)$ is defined in the boundary value problem(BVP)

$$w(\gamma+1, y) = \sum_{i=-k}^{k} \beta_i(\gamma, y+i) - GR^{-1}G^*, z(\gamma, y) \in [0,1,\dots,] \times Z \quad (2.14)$$

within the condition $w(0, y) = \vartheta(y), z(T, y) = 0, y \in Z$.

Quadratic functional in Optimization

A function in a quadratic functional is called a control function if the sequence $y \to u(\gamma, y)$ is square summable for each fixed $\gamma \in [0,1,\dots,T]$. $w(\gamma, y)$ is the corresponding trajectory with initial condition $w(0, y) = \vartheta(y), y \in Z, \vartheta \in l^2(u\ )$ whenever $T > 1$ and $y \in Z$. We shall call $w(\gamma, y)$ the solution of (2.7) if given an initial function $\vartheta \in l_2(U)$ and $u(\gamma, y) \in \beta(\alpha)$ and the function, $w: T \times Z \to V$ s.t. $w(\gamma, y) \in l^2(u)$ satisfies equation (2.7) and the initial condition.

Most times optimal control problems are set to find the control function that minimizes the cost functional

$$J(u) = \sum_{w=1}^{T} \sum_{y=z}^{T-1} [\langle Pw(\gamma, y), w(\gamma, y) \rangle + \langle Ru(\gamma-1, y), u(\gamma-1, y) \rangle] \quad (3.1)$$

Where $w(\gamma, y)$ is the solution as defined earlier corresponding to the initial condition and control u. We remark here that $P: U \to U, R: V \to V$ are self adjoint operators and positive definite. We present some theorems and proof that will show that an optimization problem is solvable.

Proof of Main theorem A:

Consider the system in (2.7) in the operator form

$$\kappa_{\gamma+1} = \beta\kappa_\gamma + G\alpha_\gamma, \gamma \in [0,1,\dots,T] \quad (3.2)$$

where β and G are operators. Let $\beta_T(u)$ and $\beta_T(\alpha)$ be function spaces defined on $\{0,1,...,T\}$ with values in the spaces $l_2(G), l_2(V)$respectively such that

$\beta_T(U) = (l_2(U))^{T+1}, \beta_T(V) = (l_2(V))^{T+1}$ and $l: \beta(V) \to \beta_T(U)$,producing the mapping.

$$(L\mu)_\gamma = G\mu_{\gamma-1} + \beta G\mu_{\gamma-2} + \dots + \beta^{\gamma-2}G\mu_1 + \beta^{\gamma-1}G\mu_0, (L\mu)_0 = 0, \quad (3.3)$$

where $\mu = (\mu_0, \mu_1, ..., \mu_T) \in \beta_T(U), \beta_T^0(U)$ □ $\beta_T(U)$ containing the function with zero values at $\gamma = 0$.

Next, we show that by using the form $w = L\alpha + \omega$ (3.4)

the solution of (3.1) can be presented analytically as follows,

$$\gamma = 0, \text{K}_0 = w(0, y) = (L\alpha + \omega)_0 = \vartheta(y)$$
$$\gamma = 1, \text{K}_1 = w(1, y) = (L\alpha + \omega)_1 = G\alpha_0 + \beta\vartheta$$
$$\gamma = 2, \text{K}_2 = w(2, y) = (L\alpha + \omega)_2 = G\alpha_1 + \beta G\alpha_0 + \beta^2\vartheta$$
$$\vdots$$
$$\gamma = T, \text{K}_T = w(T, y) = (L\alpha + \omega)_T = G\alpha_{T-1} + G\beta\alpha_{T-2} + \dots + \beta^{T-1}Gv_0 + \beta^T\vartheta$$

This with (3.3) proves (3.4).

Rewriting the cost functional (3.1) in the operator form as

$$\langle Pw, w\rangle + \langle Ru, u\rangle = \langle Pw, w\rangle + \langle R\alpha, \alpha\rangle$$
$$= \langle P(L\alpha + \omega), (L\alpha + \omega)\rangle + \langle R\alpha, \alpha\rangle$$
$$= \langle PL\alpha, L\alpha\rangle + \langle PL\alpha, w\rangle + \langle Pw, L\alpha\rangle + \langle Pw, w\rangle + \langle R\alpha, \alpha\rangle$$
$$= \langle L * PL\alpha, \alpha\rangle + \langle PL\alpha, w\rangle + \langle L * Pw, \alpha\rangle + \langle Pw, w\rangle + \langle R\alpha, \alpha\rangle$$
$$== \langle (L * PL + R)\alpha, \alpha\rangle + \langle PL\alpha, w\rangle + \langle L * Pw, \alpha\rangle + \langle Pw, w\rangle + \langle R\alpha, \alpha\rangle$$
$$= \langle (R + L * PL)\alpha, \alpha\rangle + 2\langle L * Pw, \alpha\rangle + \langle Pw, w\rangle + \langle R\alpha, \alpha\rangle$$

$$= \langle (R + L * PL)v, v\rangle + 2\langle L * P\omega, v\rangle + \langle P\omega, \omega\rangle$$

Hence,

$$J(u) = \langle (R + L * PL)v, v\rangle_\beta + 2\langle L * P\omega, v\rangle_\beta + \langle P\omega, \omega\rangle_\beta \quad (3.5)$$

where $w = (\vartheta, \beta\vartheta, ... (\beta)^T\vartheta \in G_T(U))$ and the operators $P: G_T(U) \to G_T(U)$ and $R: G_T(V) \to (G_T(V))$ are clearly given in the manner

$P(w)(\gamma, y) = Pw(\gamma, y), \gamma = 0,1, \dots, T, y \in Z, (Ru)(\gamma, y) = Ru(\gamma - 1, y), \gamma \neq 0.$

Using Freshet derivative, we obtain the derivative of (3.5) as

$$\frac{\partial J(\alpha)}{\partial \alpha} = 2(R + L * PL)\alpha + 2L * Pw\text{=}0$$

which produces the solution, $\alpha = -(R + L * PL)^{-1}L * Pw$ and

$$\alpha^0 = -(R + L * PL)^{-1}L * Pw. \quad (3.6)$$

Finally, we prove the optimality of the solution obtained by checking the inequality $J(\alpha) \geq J(\alpha^0)$ for any $v \in G_T(V)$. This can be ascertained by first showing the equality of the difference of the function J evaluated at α and α^0. If true, it shows that $\alpha^0$is the optimal control in the initial problem. Let $\zeta = (R + L * PL)$, to give

$$J(\alpha) \geq J(\alpha^0)\alpha^0 = \langle \zeta\alpha, \alpha\rangle + 2\langle L * Pw, \alpha\rangle - \langle L * Pw, L * Pw\rangle$$
$$= \langle \zeta\alpha + \zeta\zeta^{-1}L * Pw, \alpha\rangle + \langle L * Pw, \alpha + \zeta^{-1}L * Pw\rangle$$
$$= \langle \zeta(\alpha - \alpha^0), \alpha\rangle + \langle L * Pw, \alpha - \alpha^0\rangle$$
$$= \langle \zeta(\alpha - \alpha^0), \alpha\rangle - \langle \zeta\alpha^0, (\alpha - \alpha^0)\rangle$$
$$=\langle \zeta(\alpha - \alpha^0), \alpha - \alpha^0\rangle$$

Next, since the operators P and R are positive definite and operator ζ positive, the inverse of ζ will be positive. We conclude this proof by saying that since

$J(\alpha) \geq J(\alpha^0) = \langle (R + L * PL)\alpha \geq (R + L * PL)\alpha^0\rangle_G > 0$ for any $\alpha \in G_T(V), \alpha \neq \alpha^0$, then $\alpha^0$is a solution orproblem

The Uniqueness of Solution to the Optimal Control Problem

We shall discuss and prove that the solution, so obtained from our problem will be unique. Two approaches shall be adopted in presenting the proof of the uniqueness of our solution and these are given in the proof of the main theorem B, as follows.

Proof of Main theorem B

Let us also call $u^0(\gamma, y) = -R^{-1}G^*z(\gamma, y)$ the optimal feedback control. To show its uniqueness, assume $w(\gamma, y), z(\gamma, y)$ are solutions of the BVP given in (2.14) with $\gamma \in [0,1,2, \dots, T), y \in$ Z. Next consider, $u(\gamma, y) = -R^{-1}G^*z(\gamma, y)\gamma \in [0,1, \dots, T], \gamma \in [0,1, \dots, T], y \in Z_T$

Then from equation (2.12) – (2.14), we can obtain

$$\kappa_{\gamma+1} = \beta\kappa_\gamma - GR^{-1}G^*\vartheta_\gamma \quad (4.1)$$

$$\vartheta_\gamma = \beta^*\vartheta_{\gamma+1} + PK_{\gamma+1,}w_0 = \vartheta, \vartheta_T = 0 \quad (4.2)$$

Hence,

$K_\gamma = \beta^\gamma\vartheta - \sum_{i=0}^{\gamma-`} GR^{-1}G^*\beta^*\vartheta_{\gamma-1-i}, \gamma_0 = 0, y = \vartheta, \dots, T, \vartheta_T = 0$

$K_\gamma = \beta^\gamma\vartheta - \sum_{i=0}^{\gamma-1} GR^{-1}G^*\vartheta_{\gamma-1-i}, \gamma \in [0,1, \dots, T]$

$$\vartheta_\gamma = \beta^\gamma\vartheta - \sum_{i=0}^{T-\gamma} \beta^{*i}PK_{\gamma+1}, \bar{\alpha}_\gamma = R^{-1}G^*\vartheta_\gamma \quad (4.3)$$

where $\bar{\alpha}_\gamma(y) = \check{u}(\gamma, y)$.

Based on equations (2.12) and (2.13), where $\bar{\alpha}$ comprises of $\bar{\alpha} = (\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_T) \in G_T(V)$, we can write it in the form

$$\bar{\alpha} = -R^{-1}L * P_{\text{Ҡ}}, \text{Ҡ} = (\text{K}_0, \text{K}_1, \dots, \text{K}_T)$$

Hence,

$$R\bar{\alpha} = -L^*P\text{Ҡ} \quad (4.4)$$

Similarly, we know that

$K = w - LR^{-1}G^*w\alpha$ an $= w + L\bar{\alpha}, w = (w_0, w_1, \dots, w_n) \in G_T(U))$.

Thus,

$$R\bar{\alpha} = -L^*P\text{Ҡ} + L^*Pw - L^*Pw = L^*P(w - \text{Ҡ}) - L^*Pw = -L^*PL\bar{\alpha} - L^*Pw$$

and

$$\bar{\alpha} = -(R + L^*PL)^{-1}L^*Pw.$$

This follows that $\bar{\alpha}$ coincides with α^0 defined by equation (2.12). Therefore,

$$\check{u}(\gamma, y) = u^0(\gamma, y) = R^{-1}G^*(\gamma, y)$$

is the optimal control problem .

Another approach through which we can locate the optimal control of an optimization problem is enshrined in the following theorem statement below.

The optimal control of the problem given by

$$MinJ(u) = \sum_{\gamma=1}^{T}\sum_{y=z}\langle Pw(\gamma, y), w(\gamma, y)\rangle + \langle Ru(\gamma, y), u(\gamma, y)\rangle$$

Subject to $w(\gamma+1, y) = \beta_0 w(\gamma, y) + \beta_1 w(\gamma, y+1) + \beta_2 w(\gamma, y-1)$,

with initial and boundary conditions

$$w(0, y) = \vartheta(y), y \in Z_+\backslash\{0\}$$
$$w(\gamma, 0) = \psi(\gamma) = u_\gamma, \gamma \in [0,1, \dots, T-1]$$

is

$$u_\gamma^0 = -R^{-1}\beta_\gamma^* z^0(\gamma, 0), \gamma \in [0,1, \dots, T-1]$$

Where $z(\gamma, 0)$ is obtained from the system of equations.

$$z(\gamma, y) = \beta_0^* z(\gamma+1, y) + \beta_1^* z(\gamma+1, y+1) + \beta_2^* z(\gamma+1, y-1) + Pw(\gamma+1, y)$$
$$w(\gamma+1, y) = \beta_0 w(\gamma, y) + \beta_1 w(\gamma, y+1) + \beta_2 w(\gamma, y-1) - \beta_2 R^{-1}\beta_2^* w(\gamma, y)$$

with boundary conditions

$$w(0, y) = \vartheta(y), z(T, y) = 0$$

Proof: In our previous section, we showed the solvability of the optimization problem, and its optimal feedback control as

$$\alpha_0 = -(R + L^*PL)^{-1}L^*Pw.$$

Multiply through by $(R + L^*PL)$ and assume Ķ as

$$Ķ = L\alpha + w$$

to produce

$$\alpha_0 = -R^{-1}L^*PĶ_0$$

Recall that the operator L^* is conjugate to L and can be presented as $L^* = G^*\sigma$. Considering the map $G^*: l_2(U) \to V$ as

$$(G_{u,\in})_{l_2(U)} = \langle \beta_2 u, \alpha_1\rangle + \langle 0, \alpha_2\rangle + \dots + \langle 0, \alpha_n\rangle$$
$$= \langle u, \beta_2^*\alpha_1\rangle + \langle 0, \alpha_2\rangle + \dots + \langle 0, \alpha_n\rangle$$
$$= \langle u, \beta_2^*\alpha_1 + 0.\alpha_2 + \dots + 0.\alpha_n\rangle$$

Thus, $G^*: (\alpha_1, \alpha_2, \dots, \alpha_n) \to \beta_2^*\alpha_1$

This indicates that the operator σ is defined as

$$(\sigma\phi)_\gamma = \phi_{\gamma+1} + \beta^*\phi_{\gamma+2} + \beta^{*2}\phi_{\gamma+3} + \cdots, \phi_0 = 0, \gamma \in [0,1, \dots, T-1]$$

where the operator β^* is

$$\beta^*: (\in_1, \in_2, \in_3, \dots) \to (\beta_0^* \in_1 + \beta_1^* \in_2\ \beta_2^* \in_1 + \beta_0^* \in_2 + \beta_1^* \in_3 + \dots,)$$

Hence, we have,

$$\alpha_\gamma^0 = -R^{-1}G(PĶ_{\gamma+1}^0 + \beta^*Ķ_{\gamma+2}^0 + \cdots), \gamma \in [0,1, \dots, T-1] \quad (4.5)$$

Now, let $z_\gamma = (PĶ_{\gamma+1}^0 + \beta^*Ķ_{\gamma+2}^0 + \cdots), \gamma \in [0,1, \dots, T-1]$, then we can obviously see that z satisfies

$$z_\gamma = \beta^* z_{\gamma+1} + PĶ_{\gamma+1}^0, \gamma \in [0,1, \dots, T-1] \quad (4.6)$$

Which represents the conjugate one.

As long as equations (2.7) and (3.4) are solvable for any, then the conjugate system is equally solvable. A unique solution z^0 for any κ^0 is contained in equation (4.4). Since we can easily show that (4.6) can be turned to the form of (3.13), hence from (4.5), we have,

$$\alpha_\gamma^0 = -R^{-1}G^* z_\gamma^0$$

Accordingly, we have,

$$u_\gamma^0 = -R^{-1}\beta_2^* z^0(\gamma, 0), \gamma = 0,1,2, \dots, T-1.$$

This completes the proof of our theorem.

Conclusion

The study so far has given us an opportunity to examine the problem of 2 - dimensional Optimization problems / linear systems, exposed the usefulness of this area of optimization and the need to solve, provide solutions that are unique and stand the test of time. In the course of the study, we have been able to provide the proof for the solvability of a 2 - dimensional optimization problem. This platform has enabled us to equally show that the solutions generated from these problems are unique. Uniqueness of solutions enhances application. To ensure productivity when these results are applied, we also test the optimality of our solutions by checking if the inequality $J(\alpha) \geq J(\alpha^0)$ for any $\alpha \in G_T(V)$ is satisfied. These results guarantee the user of the 2 - dimensional optimization models a place in the industrial setting, thus erases the anxiety often nursed users of newly developed optimization models.

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